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Fully nonlinear gravity-capillary solitary waves in a two-fluid system of finite depth

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Abstract. Large-amplitude waves at the interface between two laminar immisible inviscid streams of different densities and velocites, bounded together in a straight infinite channel are studied, when surface tension and gravity are both present. A long-wave approximation is used to develop a theory for fully nonlinear interfacial waves allowing amplitudes as large as the channel thickness. The result is a set of evolution equations for the interfacial shape and the velocity jump across it. Traveling waves of permanent form are studied and it is shown that solitary waves are possible for a range of physical parameters. All solitary waves can be expressed implicitly in terms of incomplete elliptic integrals of the third kind. When the upper layer has zero density, two explicit solitary-wave solutions have been found whose amplitudes are equal to $h/4$ or $h/9$, where $2h$ is the channel thickness. In the absence of gravity solitary waves are not possible but periodic ones are. Numerically constructed solitary waves are given for representative physical parameters.

Key words: two-fluids, surface tension, solitary waves

1. Introduction

Nonlinear waves propagating on the surface of fluid of finite or infinite depth have been studied extensively. Gravity waves were studied by Stokes [1], Schwartz [2] and Longuet-Higgings [3] and exact solutions for capillary waves were found by Crapper [4] and Kinnersley [5]. Gravity-capillary waves in irrotational fluid were considered by Schwartz and Vanden-Broeck [6], Chen and Saffman [7], Hogan [8] and Hunter and Vanden-Broeck [9]. When vorticity is also present, calculations for gravity-capillary can be found in Kang and Vanden-Broeck [10] and references therein.

When two fluids are present, the additional effects of Kelvin–Helmholtz instability need to be included. Gravity waves in this case have been modelled by Liska, Margolin and Wendroff [11], and Choi and Camassa [12] among others. In particular, the latter study contains the Kortweg-de Vries and Intermediate Long Wave equations as special cases. The effect of Kelvin–Helmholtz instability, when a second fluid is introduced, is likely to render these results unstable to short waves.

The purpose of the present work is to derive a two-fluid model when surface tension and gravity are present, and at the same time allowing tangential slip at the interface making it a vortex sheet. The surface tension is a physical regularization of the system and we are concerned with the construction of solitary waves for a range of parameters. These solutions (along with finite-length waves not included here) add gravity to the study of Kinnersley [5]. No exact solutions are possible in general when gravity is present, but our fully nonlinear

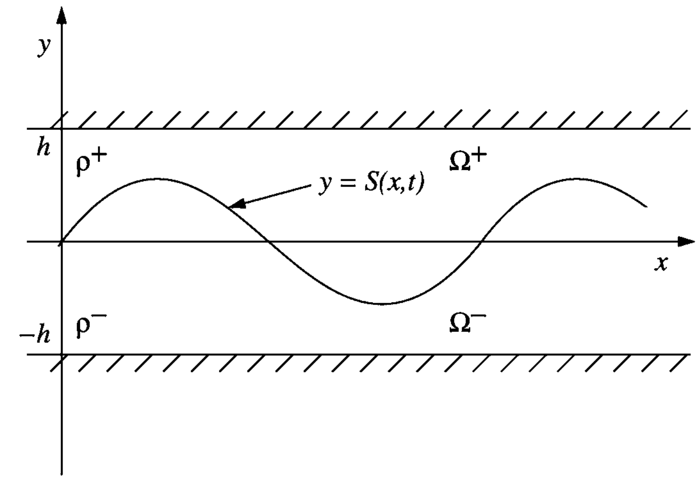

Figure 1. Two-layer fluid system.

long-wave model supports a class of solitary waves expressible in terms of elliptic integrals. Some explicit solutions are also found.

2. Formulation and governing equations

Consider the evolution of interfacial waves when both gravity and surface tension act. Two inviscid incompressible fluids of infinite horizontal extent, are bounded in a channel with straight, horizontal parallel walls. The channel height is $2h$ and initially the layers have equal thickness. Denote upper and lower fluid quantities by superscipts $+$ and $-$, respectively; the densities are ρ^+ and ρ^-. The undisturbed interface is at $y = 0$ and at later times it is given by $y = S(x, t)$, where x is the horizontal coordinate and t is time (see Figure 1).

The flow is taken to be irrotational away from the interface (in what follows we allow the interface to be a vortex sheet). The governing equations are a Laplace equation for the fluid potential, zero normal velocity at solid surfaces, a kinematic condition and a Bernoulli equation at the interface.

If l is a typical horizontal length scale (*e.g.* a wavelength), and c^* a typical velocity, the problem is made dimensionless by introducing the following variables (subscripts 1 are subsequently dropped from the dimensionless equations):

$$x_1 = \frac{x}{l}, \quad y_1 = \frac{y}{h}, \quad t_1 = \frac{c^*}{2l}t, \quad S_1 = \frac{S}{h}, \quad \phi_1^{\pm} = \frac{\phi^{\pm}}{c^* l}, \quad p_1^{\pm} = \frac{2}{\rho^-(c^*)^2}p^{\pm}. \tag{1}$$

The dimensionless system to be addressed becomes

$$\varepsilon^2\phi^{\pm}_{xx} + \phi^{\pm}_{yy} = 0 \quad \text{in} \quad \Omega^{\pm}, \tag{2}$$

$$\phi^{\pm}_y = 0 \quad \text{on} \quad y = \pm 1, \tag{3}$$

$$\varepsilon^2\left(\frac{1}{2}S_t + \phi^{\pm}_x S_x\right) = \phi^{\pm}_y \quad \text{on} \quad y = S, \tag{4}$$

$$\phi_t^- + \left[\left(\phi_x^-\right)^2 + \frac{1}{\varepsilon^2}\left(\phi_y^-\right)^2\right] + \frac{1}{F}(1-\rho)S$$
$$-\rho\phi_t^+ - \rho\left[\left(\phi_x^+\right)^2 + \frac{1}{\varepsilon^2}\left(\phi_y^+\right)^2\right] = \tilde{\sigma}\varepsilon\frac{S_{xx}}{\left(1+\varepsilon^2 S_x^2\right)^{\frac{3}{2}}} \quad \text{on} \quad y = S, \tag{5}$$

where the dimensionless parameters

$$\varepsilon = \frac{h}{l}, \quad \rho = \frac{\rho^+}{\rho^-}, \quad F = \frac{(c^*)^2}{2gh}, \quad \tilde{\sigma} = \frac{2\sigma^*}{l\rho^-(c^*)^2} \tag{10}$$

are a shallowness parameter, density ratio, Froude number and a surface-tension parameter. The right-hand side of (5) is the contribution to the pressure jump across the interface due to surface tension.

In what follows, canonical equations are derived in the limit $\varepsilon \to 0$ with both gravity and surface tension retained. In the long-wave analysis the curvature of the interface is small relative to its amplitude, and in order to allow for surface-tension effects to enter and compete with gravity, the distinguished limit,

$$\tilde{\sigma} = \frac{\sigma}{\varepsilon} \tag{11}$$

is considered, with σ an order-one parameter. Using this, Equation (5) becomes

$$\varepsilon^2\left[\phi_t^- - \rho\phi_t^+ + (\phi_x^-)^2 - \rho(\phi_x^+)^2\right] + (\phi_y^-)^2 - \rho(\phi_y^+)^2 + \frac{\varepsilon^2}{F}(1-\rho)S = \frac{\sigma\varepsilon^2 S_{xx}}{(1+\varepsilon^2 S_x^2)^{3/2}}. \tag{12}$$

The problem stated above is exact and results from the chosen non-dimensionalization; for example, by setting $\varepsilon = 1$, (2–4), (12) become the equations for a non-slender two-fluid system in a channel. In what follows we study strongly nonlinear solutions valid in the limit $\varepsilon \to 0$.

3. Derivation of the nonlinear evolution equations

In the limit $\varepsilon \to 0$, the only small parameter appearing in the governing Equations (2–4), (12) is ε^2 and we assume the following asymptotic expansions,

$$\phi^\pm = \phi_0^\pm + \varepsilon^2\phi_1^\pm + \varepsilon^4\phi_2^\pm + \cdots, \quad S = S_0 + \varepsilon^2 S_1 + \varepsilon^4 S_2 + \cdots. \tag{13}$$

When (13) are substituted in (2–4), (12) and successive orders of ε^2 are equated, the following problems emerge:

$$\phi_{iyy}^\pm = a_i^\pm(x, y, t), \tag{14}$$

$$\phi_{iy}^\pm = 0 \quad \text{on} \quad y = \pm 1, \tag{15}$$

$$\phi_{iy}^\pm = c_i^\pm(x, y, t) \quad \text{on} \quad y = S_0(x, t), \tag{16}$$

where $i = 0, 1, 2, \ldots$. The functions a_i and c_i force the linear operators on the left-hand side and contain information from the previous stage. The first few of these functions are

$$a_0^{\pm} = 0, \quad a_i^{\pm}(x, y, t) = -\phi^{\pm}_{i-1,xx}, \quad i = 1, 2, \ldots \tag{17}$$

$$c_0^{\pm} = 0, \quad c_1^{\pm} = \tfrac{1}{2}S_{0t} + S_{0x}\phi^{\pm}_{0x}, \quad c_2^{\pm} = \tfrac{1}{2}S_{1t} + S_{1x}\phi^{\pm}_{0x} + S_{0x}\phi^{\pm}_{1x} + S_1\phi^{\pm}_{0xx}, \tag{18}$$

where the potential functions and their derivatives in Equation (18) are evaluated at $(x, y = S_0(x,t), t)$. For each i the problem is a linear boundary-value one, which has a solution (unique up to a constant) if and only if the following compatibility condition is satisfied (see [13] for a situation involving elliptic operators on the left):

$$\langle a_i^- \rangle = \int_0^1 \mathrm{d}x \int_{-1}^{S_0} a_i^-(x, y, t)\,\mathrm{d}y = \int_0^1 c_i^-(x, t)\,\mathrm{d}x, \tag{19}$$

$$\langle a_i^+ \rangle = \int_0^1 \mathrm{d}x \int_{S_0}^{1} a_i^+(x, y, t)\,\mathrm{d}y = -\int_0^1 c_i^+(x, t)\,\mathrm{d}x. \tag{20}$$

The above conditions assume (without loss of generality) periodic solutions with period one in the x-direction. The leading-order problem can be easily solved to give solutions that are independent of y

$$\phi_0^+ = \Phi_0^+(x, t) \quad \text{in} \quad \Omega_+, \qquad \phi_0^- = \Phi_0^-(x, t) \quad \text{in} \quad \Omega_-. \tag{21}$$

At the next order $i = 1$, the compatibility conditions are

$$\int_0^1 \left[\frac{1}{2}S_{0t} + \left((S_0+1)\Phi^-_{0x}\right)_x\right]\mathrm{d}x = 0, \qquad \int_0^1 \left[\frac{1}{2}S_{0t} + \left((S_0-1)\Phi^+_{0x}\right)_x\right]\mathrm{d}x = 0. \tag{22}$$

Since the integrals in (22) are valid for all times, the integrands must vanish giving the equations

$$\tfrac{1}{2}S_{0t} + \left((S_0+1)\Phi^-_{0x}\right)_x = 0, \tag{23}$$

$$\tfrac{1}{2}S_{0t} + \left((S_0-1)\Phi^+_{0x}\right)_x = 0. \tag{24}$$

The Bernoulli equation (12) is satisfied identically at order ε^0, and at order ε^2 yields

$$\Phi^-_{0t} - \rho\Phi^+_{0t} + (\Phi^-_{0x})^2 - \rho(\Phi^+_{0x})^2 + \frac{1-\rho}{F}S_0 = \sigma S_{0xx}. \tag{25}$$

Equations (23–25) are three equations for the three unknown functions Φ_0^+, Φ_0^- and S_0. They can be reduced to a set of two coupled nonlinear partial differential equations by defining new variables Φ and V by

$$\Phi = \tfrac{1}{2}(\Phi_0^+ + \Phi_0^-), \qquad V = \tfrac{1}{2}(\Phi_0^+ - \Phi_0^-). \tag{26}$$

Addition and subtraction of (23) and (24) and integration with respect to x in the latter instance gives

$$S_{0t} + 2(\Phi_x S_0 - V_x)_x = 0, \qquad \Phi_x = S_0 V_x + \chi(t), \tag{27}$$

where the function $\chi(t)$ is a result of the x integration. The value of $\chi(t)$ can be found by considering the unperturbed flow at large $|x|$. Assuming that far away the interface is flat, we have $S_0(\pm\infty, t) = 0$; Φ_x is the average of the undisturbed fluid velocities in the two layers and if the fluids are at rest far away, it follows that $\chi(t) = 0$. The general case has $\Phi_x \neq 0$ and thus $\chi \equiv \chi(t)$. The case $\chi =$ const. corresponds to uniform inviscid streams while any time oscillatory far fields, for example, give rise to a time dependence. Eliminating Φ, we see that (27) becomes

$$S_{0t} + 2\chi S_{0x} + 2\left(S_0^2 V_x\right)_x = 2V_{xx}.$$

A second equation is obtained by differentiation of (25) with respect to x and elimination of $\phi_0^{\pm}$ in terms of the new variables (26). The resulting evolution equations are

$$S_t + 2\chi S_x + 2(S^2 W)_x = 2W_x, \tag{28}$$

$$(W - \alpha SW)_t + 2\chi(W - \alpha SW)_x - (\alpha S^2 W^2 - 2SW^2 + \alpha W^2)_x = \frac{\alpha}{F} S_x - \frac{\sigma}{1+\rho} S_{xxx}, \tag{29}$$

where $W = V_x$, and the subscript zero has been dropped from S_0. If we change to the inertial frame

$$(x, t) \to \left(x - 2\int^t \chi(t')\,\mathrm{d}t', t \right),$$

we can remove χ from the problem. In what follows, then, we study Equations (28) and (29) with $\chi(t) \equiv 0$. The parameter α is the Attwood ratio defined by $\alpha = \frac{1-\rho}{1+\rho}$, and $-1 \le \alpha \le 1$. When α is positive/negative, the heavier fluid is on the bottom/top; the latter case introduces a Rayleigh-Taylor instability into the problem. The case $\alpha = 0$ is that of density matched fluids. The system (28), (29) contains various physical mechanisms of interest including Kelvin-Helmholtz instability and its modification due to surface tension and/or gravity.

The following integrals, corresponding to mass, momentum and energy, are conserved quantities of the system

$$\mathcal{I}_1 = \int S\mathrm{d}x, \qquad \mathcal{I}_2 = \int (W - \alpha SW)\mathrm{d}x, \tag{30}$$

$$\mathcal{I}_3 = \int \left[\frac{1}{2} W^2 (1 - \alpha S)(1 - S^2) - \frac{\alpha}{4F}(1 - S^2) + \frac{\sigma}{4(1+\rho)} S_x^2 \right] \mathrm{d}x. \tag{31}$$

The energy integral can be derived by starting from the exact energy of the system and applying the perturbation scheme of this section. It is easy to check *a posteriori* that Equation (29) follows from $\mathcal{I}_3$.

The case $\alpha = 1$ is particularly interesting because it yields closed-form solitary waves (see Section 4). Setting $\alpha = 1$ in (29) (note that $\chi = 0$ also) and defining a new dependent variable by $U = (1 - S)W$, we may cast the system (28), (29) into the simpler form

$$S_t - 2\,(U(1 + S))_x = 0, \qquad U_t - (U^2)_x = \frac{1}{F} S_x - \sigma S_{xxx}. \tag{32}$$

In view of the fact that explicit solitary waves exist, it would be interesting to study the system (32) for complete integrability.

We note (the calculations are not included) that the dispersion relation of the linearized version of (28), (29) coincides with the $\varepsilon \to 0$ limit of the full stability problem obtained by linearization of (2–4) and (12).

4. Nonlinear traveling waves

In this section we construct traveling-wave solutions to the system (28) and (29) with $\chi(t) \equiv 0$ as explained in Section 3. Looking for solutions of the form

$$S = S(\xi), \quad W = W(\xi), \quad \xi = x - ct$$

enables us to reduce the system (28), (29) to the ordinary differential equations:

$$-cS' + 2(S^2W)' = 2W', \tag{33}$$

$$-c(W - \alpha SW)' - (\alpha S^2W^2 - 2SW^2 + \alpha W^2)' = \frac{\alpha}{F}S' - \gamma S''', \tag{34}$$

with $'$ denoting differentiation with respect to ξ. Integration of (33), (34) yields

$$W = -\frac{1}{2}\frac{A + cS}{1 - S^2}, \tag{35}$$

$$-c(W - \alpha SW) - \alpha S^2W^2 + 2SW^2 - \alpha W^2 = \frac{\alpha}{F}S - \gamma S'' + B, \tag{36}$$

where A and B are unknown real constants. Next, multiplication of (36) by S', elimination of W from (35) and integration gives the following equation for S (see appendix for details):

$$\frac{1}{2}\gamma(S')^2 = \frac{\alpha}{2F}S^2 + BS - \frac{1}{4}\alpha c^2 S + D - \frac{1}{8}\frac{(A + c)^2(1 - \alpha)}{1 - S} - \frac{1}{8}\frac{(A - c)^2(1 + \alpha)}{1 + S}, \tag{37}$$

where D is another constant of integration. We are interested in the range $-1 < S(\xi) < 1$. It is found that for a wide range of parameters, two roots of $(S')^2 = 0$ in $-1 < S < 1$ can exist. These values define the wave maximum and minimum and solutions easily follow by quadrature. No traveling waves exist if $(S')^2 < 0$ for all $-1 < S < 1$.

4.1. Solitary waves

We look for solitary waves by setting S and its derivatives equal to zero at infinity. This implies that $-\frac{1}{2}A$ is equal to the undisturbed vortex sheet strength at infinity. In addition a double root of $(S')^2 = 0$ is at $S = 0$ (this local double root behavior requires S to tend to zero as $|\xi|$ tends to infinity) and the construction is complete if another root exists in $(-1, 1)$. For solitary waves the constants B and D can be expressed in terms of A and c by,

$$B = \frac{A}{4}(2c - \alpha A), \quad D = \frac{1}{4}(A^2 + c^2 - 2\alpha Ac), \tag{38}$$

and (37) becomes

$$\gamma(S')^2 = S^2\left[\frac{\alpha}{F} + \frac{\alpha S - 1}{2(1 - S^2)}\left\{A^2 + c^2 - 2cA\beta(S)\right\}\right]. \tag{39}$$

The function $\beta(S)$ is given by

$$\beta(S) = \frac{S - \alpha}{\alpha S - 1} \qquad \Rightarrow \qquad -1 < \beta < 1, \tag{40}$$

and the double root at $S = 0$ is clearly seen in (39). Using (40) and the fact $A^2 + c^2 - 2cA\beta(S) = (A - c\beta)^2 + c^2(1 - \beta^2) \geq 0$, we can show that $(S')^2 < 0$ whenever $\alpha \leq 0$. Physically, this says that the model does not admit solitary waves if the densities are equal or if a heavier fluid lies above a lighter one. This is expected due to the Rayleigh-Taylor instability.

In what follows, then, we consider $\alpha > 0$ and re-write (39) as

$$\gamma(S')^2 = \frac{S^2}{1-S^2} p_2(S), \tag{41}$$

$$p_2(S) = -\frac{\alpha}{F}S^2 + \frac{\alpha}{2}\left(A^2 + c^2 - \frac{2cA}{\alpha}\right)S + \left(\frac{\alpha}{F} - \frac{A^2+c^2}{2} + \alpha Ac\right). \tag{42}$$

The quadratic $p_2(S)$ plays a crucial role in the existence of solitary waves for unequal densities leading to a two-parameter family of solutions depending on A and c. The multiplying function $S^2/(1-S^2)$ in (41) is concave, symmetric about $S = 0$ and tends to plus infinity as $S = \pm 1$ from below and above, respectively. In addition, p_2 is convex since $\mathrm{d}^2 p_2/\mathrm{d}S^2 = -2\alpha/F < 0$. Four relevant possibilities emerge: (i) $p_2(S)$ has two real roots in $-1 < S < 1$ of opposite signs, (ii) $p_2(S)$ has two real roots in $-1 < S < 1$ with the same sign, (iii) $p_2(S)$ has only one real root in $-1 < S < 1$, (iv) $p_2(S)$ has no real roots in $-1 < S < 1$. Cases (ii) and (iv) preclude solitary waves, while case (i) allows two distinct solitary waves with the same propagation speed and case (iii) only one. A schematic representation of these four possibilities is given in Figure 2. Only non-negative values of S_ξ^2 can be considered and the sketches are phase diagrams. For example, traversing the part of the phase plane in Figure 2(i) from the negative root $S = S_2$ to $S = 0$, constructs half the 'left' solitary wave from its global minimum value $S_2 < 0$ at $\xi = 0$ (the origin can be fixed by the translation invariance of the equations) to its zero asymptotic value at $\xi = +\infty$; the other half follows from symmetry. A similar construction gives the positive or 'right' solitary wave by starting at $S = S_1 > 0$ at $\xi = 0$ and monotonically decreasing to the zero asymptotic value at $\xi = +\infty$. Each solitary wave can be constructed independently and this becomes useful in our analysis that casts these solutions in terms of elliptic integrals. For case (ii), it is clear from the canonical diagram in Figure 2(ii), that it is impossible to connect either of the roots S_1, S_2 with the homoclinic point $S = 0$, without traversing a region where $S_\xi^2 < 0$ - this means that no real solitary waves exist in such instances.

4.1.1. *The case $A = 0$ - zero vortex sheet strength*

When $A = 0$, Kelvin-Helmholtz instabilities are absent; in addition, these solutions can be used to obtain others with $A \neq 0$ by continuation. For $A = 0$, Equation (41) contains c^2 on its own and so constructed waves can have equal and opposite speeds. (Note also that $\alpha > 0$ as explained previously.)

Setting $A = 0$ in (42) and solving, we obtain the two roots

$$S_{1,2} = q \pm \left[q^2 - \frac{2}{\alpha}q + 1\right]^{1/2}, \quad q = \frac{1}{4}Fc^2, \tag{43}$$

and for real distinct roots we must have $q^2 - \frac{2}{\alpha}q + 1 > 0$, that is

$$q < q_2 = \frac{1-\sqrt{1-\alpha^2}}{\alpha} \quad \text{or} \quad q > q_1 = \frac{1+\sqrt{1-\alpha^2}}{\alpha}. \tag{44}$$

It is easy to establish the existence of two solitary waves for small values of q; the two roots of (43) for $0 < q \ll 1$ are

$$\begin{aligned} S_1 &= 1 - \frac{1-\alpha}{\alpha}q + O(q^2) \quad \Rightarrow \quad 0 < S_1 < 1, \\ S_2 &= -1 + \frac{1+\alpha}{\alpha}q + O(q^2) \quad \Rightarrow \quad -1 < S_2 < 0. \end{aligned} \tag{45}$$

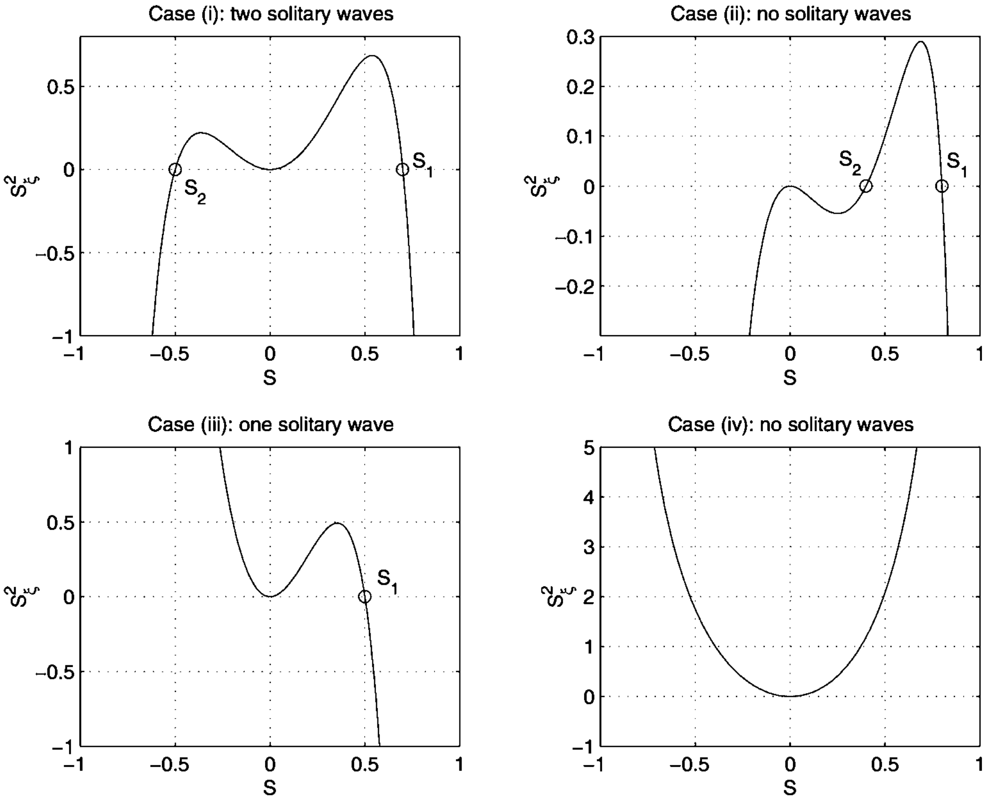

Figure 2. Schematic of the four canonical cases for solitary waves.

Continuation to larger values of q provides solitary waves with larger speeds and smaller amplitudes. The inequalities (44) define two regions in the $q-\alpha$ plane where solitary waves may exist. These are depicted as regions 1 and 2 in Figure 3 and correspond to the right and left bounds of (44), respectively. Given $\alpha > 0$ and a Froude number F, we need to find values of c for which $S_1 < 1$ and/or $-1 < S_2 < 0$.

A necessary condition for solitary waves to exist is for q and α to lie in either of regions 1 and 2 of Figure 3. The roots (43) in region 1 are

$$\begin{aligned} S_1(q) &= q + (q_1 - q)^{1/2}(q_2 - q)^{1/2}, \\ S_2(q) &= q - (q_1 - q)^{1/2}(q_2 - q)^{1/2}, \quad 0 < q < q_2 < 1. \end{aligned} \tag{46}$$

It is easy to establish that $S_1(0) = (q_1 q_2)^{1/2} = 1$, $S_1(q_2) = q_2 < 1$, $\mathrm{d}S_1/\mathrm{d}q(0) = 1 - (1/\alpha) < 0$, $\mathrm{d}S_1/\mathrm{d}q(q_2^-) = -\infty$; the root $S_1(q)$ is monotonic decreasing from 1 to q_2 and all these values admit solitary waves of positive amplitude $q_2 \le S_1(q) < 1$. Similarly, $S_2(0) = -1$, $S_2(q_2) = q_2$, $\mathrm{d}S_2/\mathrm{d}q(0) = 1 + (1/\alpha) > 0$ and $\mathrm{d}S_2/\mathrm{d}q(q_2^-) = +\infty$; the root $S_2(q)$ increases monotonically from -1 to q_2 and becomes zero at $q = \alpha/2$. Since $\frac{\alpha}{2} < q_2$, the interval $\frac{\alpha}{2} < q < q_2$ supports two positive roots both less than 1. This is case (ii) described earlier and depicted in Figure 2(ii) and so is excluded. The conclusion, then, is that two solitary waves exist (of positive and negative amplitudes, respectively) for the range

$$0 < \frac{1}{4} F c^2 \le \frac{\alpha}{2}. \tag{47}$$

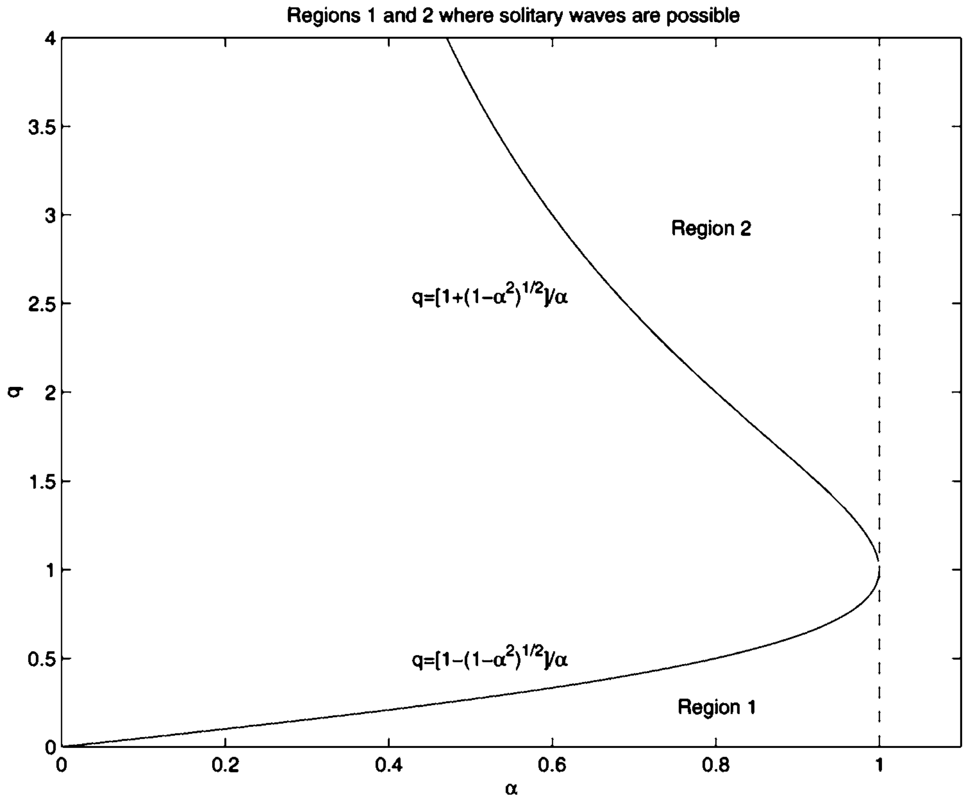

Figure 3. The regions 1 and 2 in the $q - \alpha$ plane where solitary waves may be possible.

The results presented above are summarized in Figure 4 for a typical case having $\alpha = 1/2$.

Next, we consider the possibility of solitary waves in region 2 of Figure 3. The roots of (43) are

$$\begin{aligned} S_1(q) &= q + (q - q_1)^{1/2}(q - q_2)^{1/2}, \\ S_2(q) &= q - (q - q_1)^{1/2}(q - q_2)^{1/2}, \quad q > q_1 > 1. \end{aligned} \tag{48}$$

We find $S_1(q_1) = S_2(q_1) = q_1 > 1$, and $S_1(q)$ increases monotonically as q increases with an asymptote $S_1 \sim 2q$ at large q; $S_2(q)$ is monotonic decreasing with the asymptotic behavior $S_2 \to (1/\alpha)$ as $q \to \infty$. Since $(1/\alpha) > 1$, there can be no admissible solitary waves emerging from region 2.

To summarize, we have found that for a given Froude number F and Attwood ratio $0 < \alpha < 1$, two solitary waves exist having the same speed. One wave is everywhere positive (the S_1 branch of Figure 4) and the other is everywhere negative (the S_2 branch of Figure 4). The wave speeds satisfy

$$|c| \leq \sqrt{\frac{2\alpha}{F}}, \tag{49}$$

with equality achieved when the negative solitary wave disappears and the positive one (see Figure 8 for this situation) has its smallest possible amplitude $S_{1\,\min}$, say, given by $S_{1\,\min} = \alpha$. At slower speeds, two solitary waves exist with amplitudes given by (46).

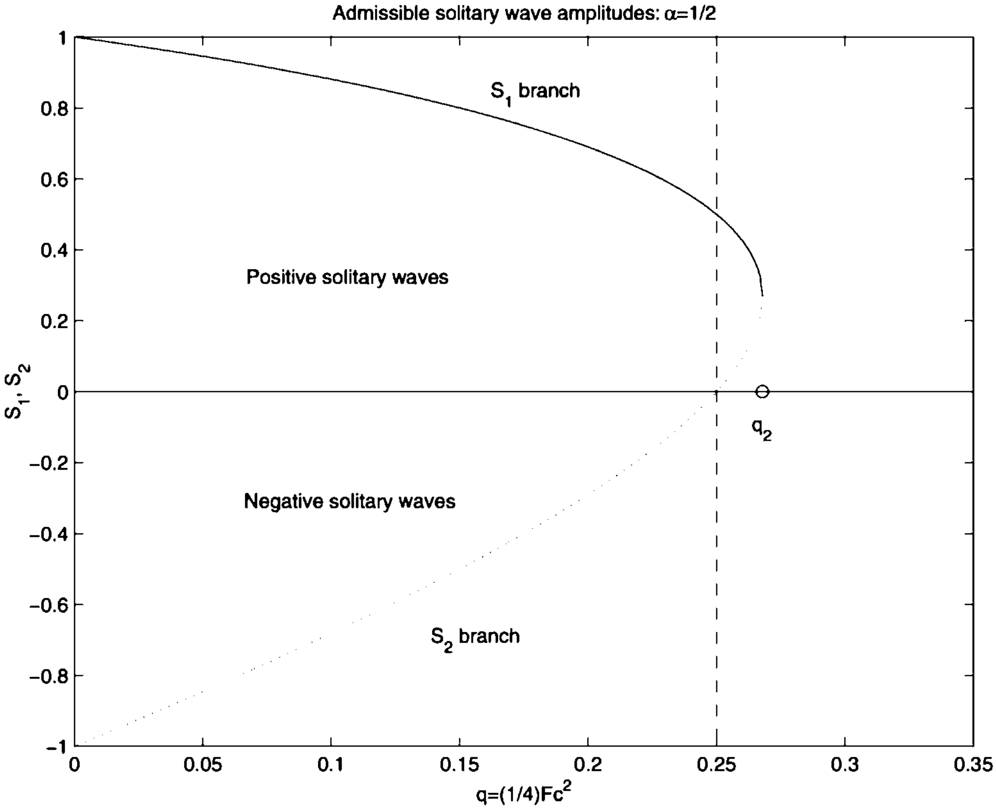

Figure 4. Admissible solitary waves in Region 1 ($0 < q < q_2$) for a typical value $\alpha = 0{\cdot}5$. No solitary waves are possible to the right of the dashed line and two distinct waves exist to the left.

Having established the existence of solitary waves for $A = 0$, we can construct solutions for $A \neq 0$ by continuation methods. For asymptotically small values of A the solitary-wave amplitudes and speeds change by order $|A|$ in a regular perturbation manner. We do not give details of such a calculation but instead use such results to guide the construction of waves numerically.

4.1.2. *Some exact solutions for $\alpha = 1$ - upper layer of zero density*

In certain cases exact solutions can be constructed depending on the wave amplitude. With $\alpha = 1$, (42) is a perfect square and it is easily shown that (41) becomes

$$\gamma(S')^2 = \frac{S^2(S+S_0)}{F(1+S)}, \quad S_0 = 1 - \frac{F}{2}(A-c)^2. \tag{50}$$

From the form of the right-hand side of (50), the only admissible solitary waves have $0 < S_0 < 1$. The resulting wave is negative everywhere and has a minimum amplitude of $-S_0$. In addition, the wave speed satisfies

$$A - \sqrt{\frac{2}{F}} < c < A + \sqrt{\frac{2}{F}}, \tag{51}$$

where A is related to the vortex sheet strength at infinity ($\lim_{|x|\to\infty} W(x) = -\frac{A}{2}$).

The solution to Equation (50) is given by (half the wave in the region $\xi > 0$ where $S' > 0$ is constructed this way, the other half by reflection)

$$\int_{-S_0}^{S}\left(\frac{t+1}{t+S_0}\right)^{1/2}\frac{\mathrm{d}t}{t}=\frac{\xi}{\sqrt{\gamma F}}. \tag{52}$$

The integral can be done exactly by making a substitution, for example $\tilde{Y}^2 = \frac{t+S_0}{t+1}$, giving

$$\frac{1+Y}{1-Y}\left(\frac{Y-\sqrt{S_0}}{Y+\sqrt{S_0}}\right)^{1/\sqrt{S_0}} = \exp\left(\frac{\xi}{\sqrt{\gamma F}}\right), \tag{53}$$

where $Y^2 = (S+S_0)/(S+1)$. It can be seen that for general values of the wave amplitude S_0, the solution must remain in implicit form. Explicit solutions are possible when $S_0 = \frac{1}{4}$ and $S_0 = \frac{1}{9}$ only, because (53) becomes a cubic or a quartic algebraic equation in Y, then. We construct the explicit solutions for these cases next.

Explicit solution for $S_0 = \frac{1}{4}$.
Manipulation of (53) leads to the following cubic equation for Y:

$$Y^3 - \frac{3}{4}Y - \frac{1}{4}\tau = 0, \quad \tau = \tanh(2\xi/\sqrt{\gamma F}). \tag{54}$$

Equation (54) has three real solutions that can be found explicitly by using Cardan's method [14, Section 1.8–4]. We choose one of these roots which satisfies conditions $\lim_{\tau\to 0} Y = 0$, $\lim_{\tau\to 1} Y = -\frac{1}{2}$. The solution is

$$Y = -\cos\left(\frac{\arccos\tau + \pi}{3}\right), \quad S(\xi) = -\frac{(1/4) - Y^2}{1-Y^2}. \tag{55}$$

We may easily check, using the fact that $\tau(\xi = 0) = 0$ and $\tau \to 1$ as $\xi \to \infty$, that $S(0) = -1/4$ and $Y^2 \to (1/4)$ as $\xi \to \infty$, thus giving the required soliton behavior at infinity. The decay to zero at infinity is exponential. The graph of this solitary wave is presented in Figure 5.

Explicit solution for $S_0 = \frac{1}{9}$.
Equation (53) in this case can be written in the form

$$Y^4 - \frac{2}{3}Y^2 - \frac{8}{27\tau}Y - \frac{1}{27} = 0, \quad \tau = \tanh(2\xi/\sqrt{\gamma F}). \tag{56}$$

Equation (56) can be solved, for instance, by Ferrari's method (see [15, Section 3.8.3] or [14, Section 1.8–6]). We found that this equation has two real roots and two complex conjugate ones and the solitary-wave solution is found explicitly by chosing one of the real roots that satisfies appropriate conditions for the solitary wave:

$$\begin{aligned} Y &= \frac{1}{3}\left\{\left(\eta+\frac{3}{2}\right)^{1/2} - \left(-\eta+\frac{3}{2}+2\left[\eta^2+\frac{3}{4}\right]^{1/2}\right)^{1/2}\right\}, \quad \eta = \left(\frac{1}{\tau^2}-1\right)^{1/3} - \frac{1}{2}, \\ S(\xi) &= -\frac{(1/9) - Y^2}{1-Y^2}. \end{aligned} \tag{57}$$

As before, we can check that $S(0) = -\frac{1}{9}$ and $S(\xi) \to 0$ as $\xi \to \infty$, which are the properties that the solitary wave must satisfy.

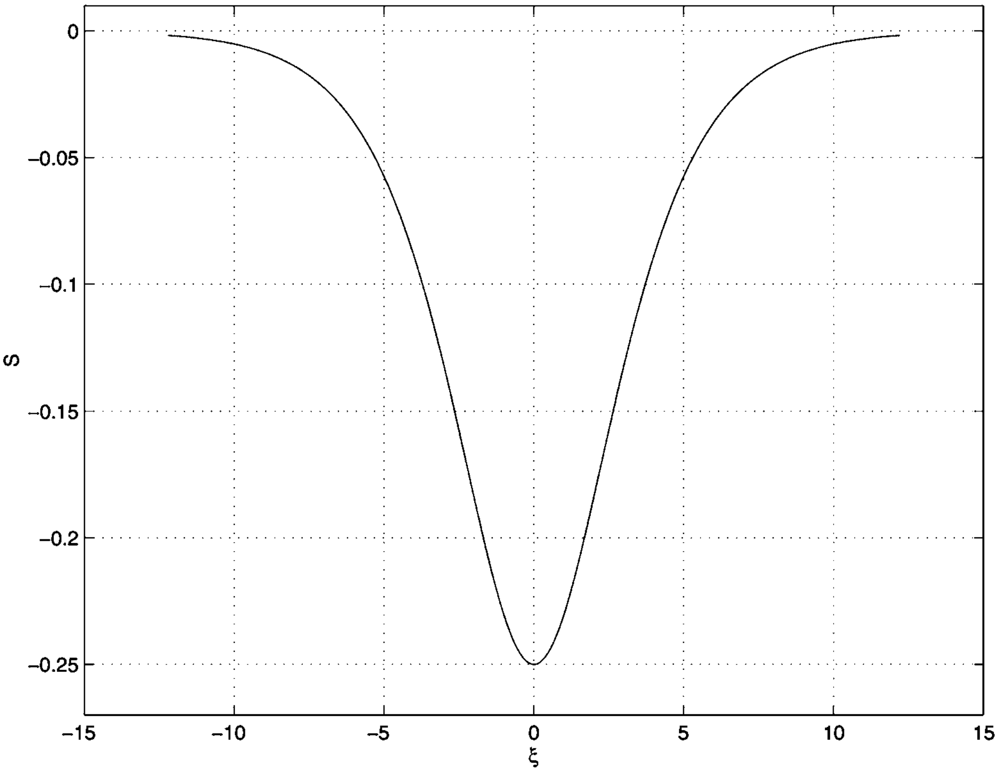

Figure 5. Solitary wave for $\alpha = 1$, $S_0 = \frac{1}{4}$.

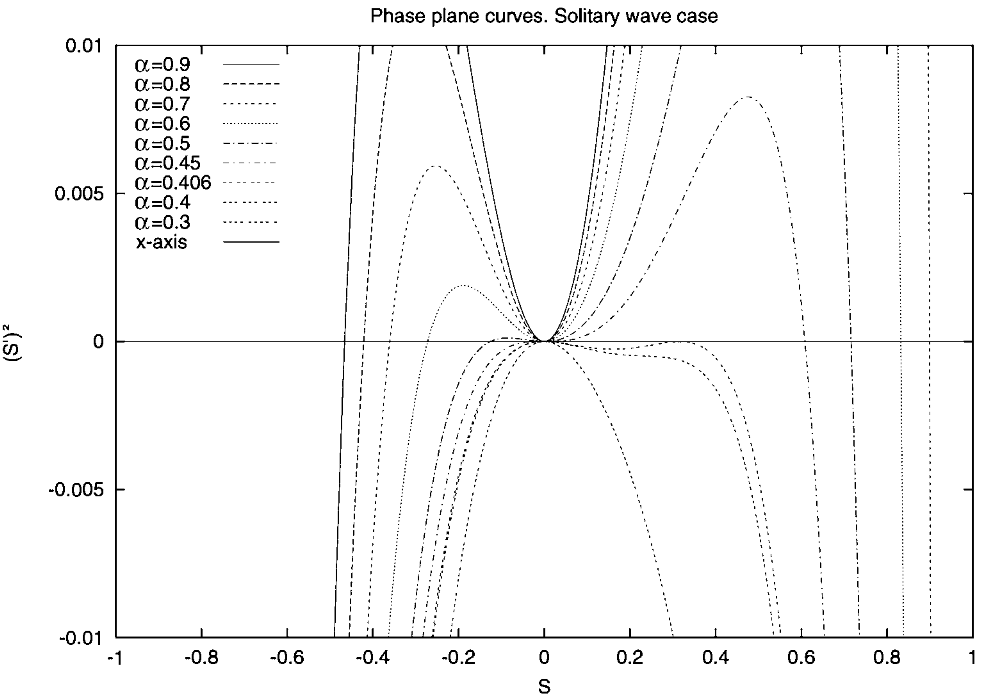

Figure 6. Phase plane curves for solitary wave case. $F = 1$, α varies.

4.1.3. *Connection with elliptic integrals*

The differential equation that gives solitary waves is (41) with (42), and as discussed previously we have

$$p_2(S) \geq 0, \quad S \in [S_2, S_1], \quad -1 < S_2 \leq 0 \leq S_1 < 1. \tag{58}$$

If constants $a_2 = -\alpha/F < 0$, $a_1 = \alpha/2(A^2 + c^2 - 2cA/\alpha)$, $a_0 = \alpha/F - (A^2 + c^2)/2 + \alpha Ac$ and $\hat{\gamma} = \sqrt{-a_2/\gamma}$, and the ratios $a = \frac{a_1}{a_2}$, $b = \frac{a_0}{a_2}$ are defined, Equation (41) becomes

$$\frac{\mathrm{d}S}{\mathrm{d}\xi} = \pm\hat{\gamma}S\sqrt{\frac{S^2 + aS + b}{S^2 - 1}} \equiv \pm F(S), \tag{59}$$

and in separated form

$$\frac{\mathrm{d}S}{\pm F(S)} = \mathrm{d}\xi \quad \text{or} \quad \frac{\mathrm{d}S}{\pm\hat{\gamma}S\sqrt{\frac{S^2+aS+b}{S^2-1}}} = \mathrm{d}\xi. \tag{60}$$

In view of (58), when the polynomial $\tilde{p}_2(S) = S^2 + aS + b$ has two real distinct roots S_1, S_2, so that $\tilde{p}_2(S) = (S - S_1)(S - S_2) \leq 0$ for $S \in [S_2, S_1]$. Hence $(S')^2$ is positive there. Note the notation introduced earlier which fixes $0 < S_1 < 1$ and $-1 < S_2 < 0$. This is the general case of interest; when a single real root is present, the analysis is similar to that given below.

Since $(\mathrm{d}S/\mathrm{d}\xi)^2$ has a double root at zero and two simple roots S_1 and S_2, we have two distinct solitary waves and each wave is treated separately. First, we integrate (60) from S_2 to S, $S_2 \leq S \leq 0$, taking the plus sign of the square root. This choice constructs the right half of the 'left' solitary wave that is negative everywhere; the other half of the wave follows by symmetry. Invoking Galilean invariance, we can shift the origin to be at the wave trough and the following implicit solution is found,

$$\xi = \int_{S_2}^{S} \frac{\mathrm{d}t}{F(t)}, \quad S_2 \leq S \leq 0. \tag{61}$$

Integration of (60) from S to S_1, $0 \leq S \leq S_1$ and use of the minus sign of the square root, constructs the right half of the 'right' solitary wave which is non-negative everywhere (translation invariance is used also to fix the origin at the wave crest); the solution is

$$\xi = -\int_{S}^{S_1} \frac{\mathrm{d}t}{F(t)}, \quad 0 \leq S \leq S_1. \tag{62}$$

Consider Equation (61) first. Introducing the polynomial

$$P_4(t) = (1 - t)(S_1 - t)(t - S_2)(t + 1), \tag{63}$$

(note that $P_4(t)$ is positive for $t \in (S_2, S_1)$, we have the solution

$$\xi = \frac{1}{\hat{\gamma}}\left\{\int_{S_2}^{S} \frac{\mathrm{d}t}{t\sqrt{P_4(t)}} - \int_{S_2}^{S} \frac{t\,\mathrm{d}t}{\sqrt{P_4(t)}}\right\} \equiv \frac{1}{\hat{\gamma}}\left\{I_1(S) - I_2(S)\right\}. \tag{64}$$

The integrals $I_1(S)$ and $I_2(S)$ can be written as a linear combination of incomplete elliptic integrals of the third kind; they are special cases of the indefinite integrals covered in items 254·11 and 254·10 on page 113 of Byrd and Friedman [16]. It is useful to introduce the following constants:

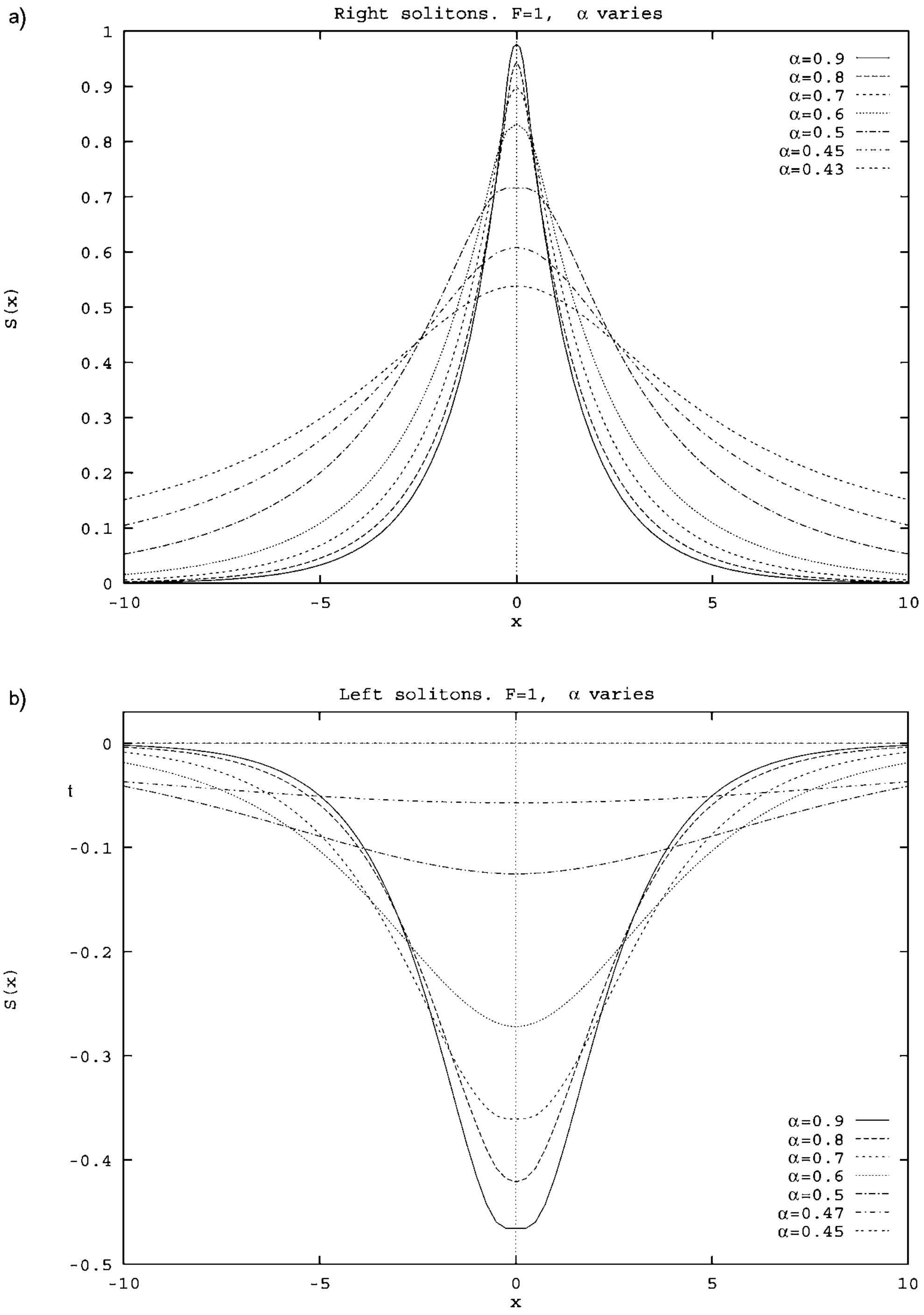

Figure 7. Solitary waves, $F = 1$, α varies: (a) left waves; (b) right waves.

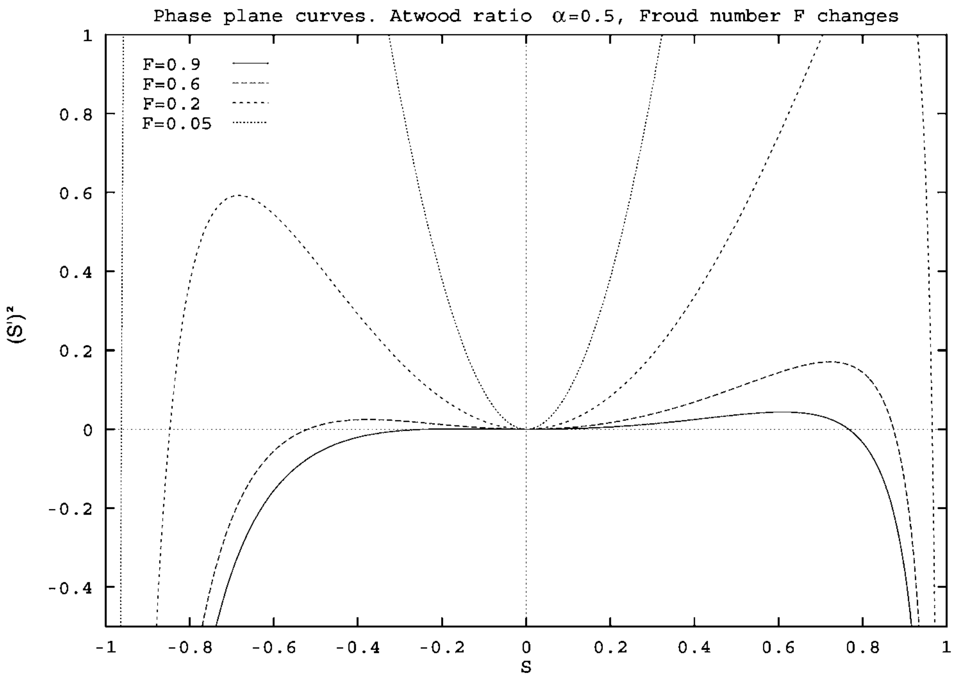

Figure 8. Phase plane curves for $\alpha = 0{\cdot}5$, $0 < F < 1$.

$$k^2 = \frac{2(S_1 - S_2)}{(1 - S_2)(S_1 + 1)}, \quad \lambda = \frac{2}{\sqrt{(1 - S_2)(S_1 + 1)}}, \quad \beta^2 = \frac{S_1 - S_2}{1 + S_1}, \quad \tilde{\beta}^2 = \frac{S_1 - S_2}{1 - S_2}, \tag{65}$$

and functions

$$\varphi(S) = \sin^{-1}\sqrt{\frac{(S_1 + 1)(S - S_2)}{(S_1 - S_2)(S + 1)}}, \quad \tilde{\varphi}(S) = \sin^{-1}\sqrt{\frac{(1 - S_2)(S_1 - S)}{(S_1 - S_2)(1 - S)}}. \tag{66}$$

The incomplete elliptic integral of the third kind is central in our solutions. This is given by (see [16]):

$$\Pi(\varphi, \beta^2, k) \equiv \int_0^S \frac{\mathrm{d}t}{(1 - \beta^2 t^2)\sqrt{(1 - t^2)(1 - k^2 t^2)}} = \int_0^\varphi \frac{\mathrm{d}\theta}{(1 - \beta^2 \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}}. \tag{67}$$

The solitary waves are given implicitly in terms of Π. The wave which is negative everywhere has

$$\frac{\hat{\gamma}}{\lambda}\xi = \frac{1 + S_2}{S_2}\Pi(\varphi, \beta^2/|S_2|, k) - (1 + S_2)\Pi(\varphi, \beta^2, k), \tag{68}$$

and the one which is positive everywhere is given by

$$\frac{\hat{\gamma}}{\lambda}\xi = \frac{S_1 - 1}{S_1}\Pi(\tilde{\varphi}, \tilde{\beta}^2/S_1, k) + (S_1 - 1)\Pi(\tilde{\varphi}, \tilde{\beta}^2, k). \tag{69}$$

5. Representative solitary waves

Equation (39) can be readily integrated. Profiles are taken to be symmetric about the origin and it is enough to calculate half of the wave. The asymptotic result (45) provides evidence regarding the existence of two solitary waves of equal speeds but different amplitudes. One wave has positive amplitude (we term this the right wave), while the other has negative amplitude (left wave). For definiteness a plus sign is taken for S' in the case of the left solitary wave and minus sign for the right one. It is convenient to use S as the variable of integration and to compute the corresponding ξ. Consequently, without loss of generality, all computed waves begin at $\xi = 0$. Solutions are obtained by quadrature by finding the ξ corresponding to the appropriate value of S. It is easy to obtain values of A and c which produce a set of right and left waves and in what follows we present solutions for the representative case

$$A = -0{\cdot}1, \qquad c = 0{\cdot}9. \tag{70}$$

The main question we address is the dependence of the solitary waves on the physical parameters F and α.

Figure 6 is a graph of the right-hand side of Equation (41) with (42) for a fixed Froude number $F = 1{\cdot}0$ and different Attwood ratios ranging from 0·9 to 0·3. A set of right and left solitary waves exists for $\alpha = 0{\cdot}9$, 0·8, 0·7, 0·6, 0·5. Note that the wave amplitudes decrease with α as does the maximum wave slope. A transition occurs between $\alpha = 0{\cdot}5$ and $\alpha = 0{\cdot}45$ when the left root coincides with the origin and the left solitary wave disappears. For $\alpha = 0{\cdot}45$ only a right wave is present. The last curve on Figure 6 has $\alpha = 0{\cdot}3$ and does not support any traveling waves since $S_\xi^2 < 0$ for all S. Representative right and left solitary waves are shown in Figures 7(a) and (b) for a range of α. For the choice of parameters (70) the amplitudes increase as the Attwood ratio increases towards unity.

The effects of Froude number are considered next. Figure 8 shows the phase plane of (41) for a set of Froude numbers and fixed $\alpha = 0{\cdot}5$. The figure covers the range of $0 < F < 1$ with the following picture emerging: at the smallest value of $F = 0{\cdot}05$ depicted, left and right solitary waves coexist which almost touch the lower and upper walls, respectively. As the Froude number increases, the amplitudes of the left and right waves move away from the walls (See Figure 9(a), (b) for left and right waves, respectively). This trend persists to higher values of F, until a smooth transition at a value of F between 1·09 and 1·1 when the left wave disappears. For $F = 1{\cdot}125$ and 1·15 right waves are still present. At higher values of F only trivial solitary waves are possible – this is seen in Figure 10 with the phase-plane curve moving below the $S_\xi^2 = 0$ axis and so precluding a right wave. The last curve has $F = 1{\cdot}25$ and is completely below the axis.

6. Conclusions

We have derived a set of nonlinear evolution equations that describe fully nonlinear gravity-capillary waves in two-fluid systems. Solitary waves have been calculated for different speeds and vortex sheet jumps. The solitary waves can be expressed in terms of incomplete elliptic integrals of the third kind. In general a positive and a negative solitary wave are present at a given speed. Admissible wave speeds satisfy $|c| \leq \sqrt{2\alpha/F}$, and when the vortex sheet strength is zero equal and opposite speeds are found. All solitary-wave solutions constructed here exhibit monotonic exponential decay at infinity. When the upper layer is absent, two elementary closed form solutions are given which correspond to negative solitary waves with minimum amplitudes equal to $-1/4$ and $-1/9$, in dimensionless terms.

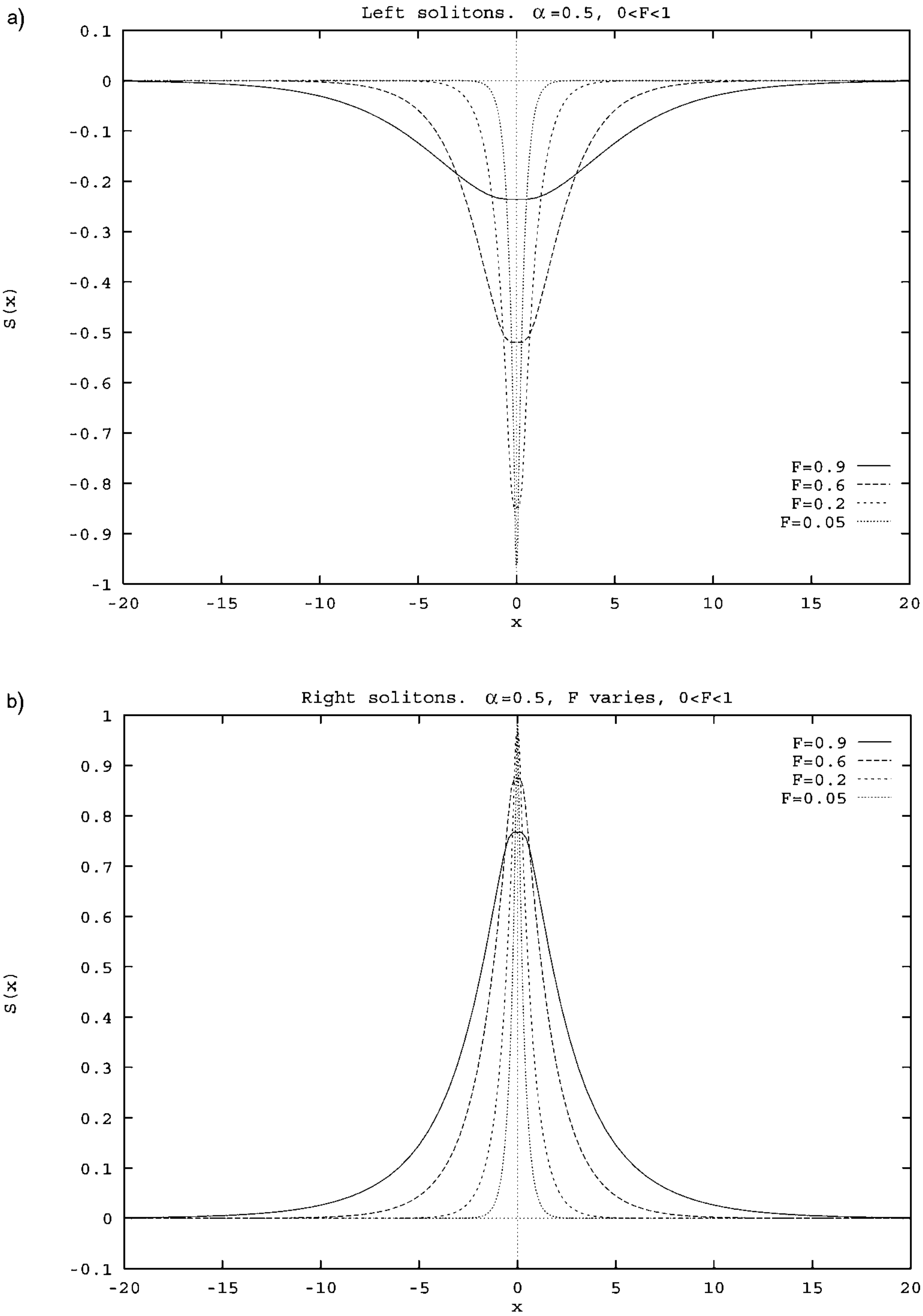

Figure 9. Solitary waves, $\alpha = 0{\cdot}5$, $0 < F < 0$: (a) left waves; (b) right waves.

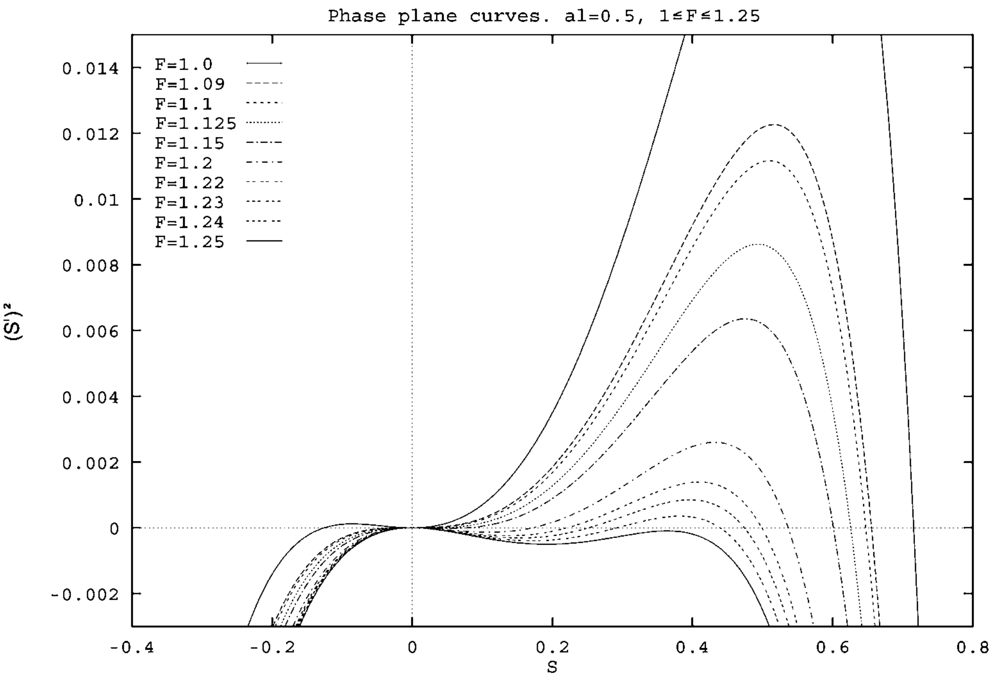

Figure 10. Phase plane curves for $\alpha = 0{\cdot}5$, $1{\cdot}0 \le F \le 1{\cdot}25$.

Appendix. Details of the derivation of (37)

Here we provide a derivation of (37), which is central in the construction of nonlinear traveling waves. Multiplication of (37) by S' and integration gives the following equation:

$$-c\int W(1-\alpha S)\mathrm{d}S - \alpha\int(1+S^2)W^2\mathrm{d}S + 2\int SW^2\mathrm{d}S = \frac{\alpha}{2F}S^2 - \frac{1}{2}\gamma S_x^2 + BS + D, \tag{A1}$$

where B and D are the same constants appearing in (37). Making use of the expression (35) for W we have the following results:

$$-c\int W(1-\alpha S)\mathrm{d}S = \frac{c}{2}\left\{c\alpha S - \frac{(c+A)(1-\alpha)}{2}\log|1-S| - \frac{(c-A)(1+\alpha)}{2}\log|1+S|\right\}, \tag{A2}$$

$$\int W^2(-\alpha S^2 + 2S - \alpha)\mathrm{d}S = -\frac{\alpha}{4}c^2 S + \frac{L_1}{4}\int\frac{\mathrm{d}S}{1-S} + \frac{L_2}{4}\int\frac{\mathrm{d}S}{(1-S)^2} + \frac{L_3}{4}\int\frac{\mathrm{d}S}{1+S} + \frac{L_4}{4}\int\frac{\mathrm{d}S}{(1+S)^2}, \tag{A3}$$

where

$$L_1 = c(A+c)(\alpha-1), \qquad L_2 = \tfrac{1}{2}(1-\alpha)(A+c)^2,$$

$$L_3 = c(1+\alpha)(c-A), \qquad L_4 = -\tfrac{1}{2}(1+\alpha)(A-c)^2.$$

The integrals in (A2) can be obtained analytically by elementary methods. Combining the results described above, we find that the logarithmic terms cancel leaving Equation (37) as the final expression.

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